IV. *Supplementary Researches on the Partition of Numbers.*

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The general formula given at the conclusion of my memoir, "Researches on the Partition of Numbers*," is somewhat different from the corresponding formula of Professor Sylvester†, and leads more directly to the actual expression for the number of partitions, in the form made use of in my memoir; to complete my former researches, I propose to explain the mode of obtaining from the formula the expression for the number of partitions.

The formula referred to is as follows, viz. if $\frac{\phi x}{fx}$ be a rational fraction, the denominator of which is made up of factors (the same or different) of the form $1-x^m$, and if a is a divisor of one or more of the indices m, and k is the number of indices of which it is a divisor, then

$$\left\{\frac{\phi x}{fx}\right\}_{[1-x^a]} = \cdots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\,\mathrm{S}\,\frac{\chi\varrho}{\varrho-x}\cdots$$

$$= \cdots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\,\frac{\theta x}{[1-x^a]}\cdots$$

where

$$\chi\varrho = \text{coeff.}\ \frac{1}{t}\ \text{in}\ t^{s-1}\,\frac{\varrho\phi(\varrho e^{-t})}{f(\varrho e^{-t})},$$

in which formula $[1-x^a]$ denotes the irreducible factor of $1-x^a$, that is, the factor which equated to zero gives the prime roots, and ϱ is a root of the equation $[1-x^a]=0$; the summation of course extends to all the roots of the equation. The index s extends from $s=1$ to $s=k$; and we have then the portion of the fraction depending on the denominator $[1-x^a]$. In the partition of numbers, we have $\phi x=1$, and the formula becomes therefore

$$\left\{\frac{1}{fx}\right\}_{[1-x^a]} = \cdots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\,\mathrm{S}\,\frac{\chi\varrho}{\varrho-x}\cdots$$

$$= \cdots + \frac{1}{\Pi(s-1)}(x\partial_x)^{s-1}\,\frac{\theta x}{[1-x^a]},$$

where

$$\chi\varrho = \text{coeff.}\ \frac{1}{t}\ \text{in}\ t^{s-1}\,\frac{\varrho}{f(\varrho e^{-t})}.$$

* Philosophical Transactions, t. cxlvi. p. 127 (1856).

† Professor Sylvester's researches are published in the Quarterly Mathematical Journal, t. i. p. 141; there are some numerical errors in his value of P (1, 2, 3, 4, 5, 6) q.

We may write
$$fx=\Pi(1-x^m),$$
where m has a given series of values the same or different. The indices not divisible by a may be represented by n, the other indices by ap, we have then
$$fx=\Pi(1-x^n)\Pi(1-x^{ap}),$$
where the number of indices ap is equal to k. Hence
$$f(\varrho e^{-t})=\Pi(1-\varrho^n e^{-nt})\Pi(1-\varrho^{ap}e^{-apt});$$
or since ϱ is a root of $[1-x^a]=0$, and therefore $\varrho^a=1$, we have
$$f(\varrho e^{-t})=\Pi(1-\varrho^n e^{-nt})\Pi(1-e^{-apt});$$
and it may be remarked that if $n\equiv\nu$ (mod. a), where $\nu<a$, then instead of ϱ^n we may write ϱ^ν, a change which may be made at once, or at the end of the process of development.

We have consequently to find
$$\chi\varrho=\text{coeff.}\ \frac{1}{t}\ \text{in}\ t^{s-1}\frac{\varrho}{\Pi(1-\varrho^n e^{-nt})\Pi(1-e^{-apt})}.$$

The development of a factor $\frac{1}{1-\varrho^n e^{-nt}}$ is at once deduced from that of $\frac{1}{1-ce^{-t}}$, and is a series of positive powers of t. The development of a factor $\frac{1}{1-e^{-apt}}$ is deduced from that of $\frac{1}{1-e^{-t}}$, and contains a term involving $\frac{1}{t}$. Hence we have
$$\frac{1}{\Pi(1-\varrho^n e^{-nt})\Pi(1-e^{-apt})}=A_{-k}\frac{1}{t^k}+A_{-(k-1)}\frac{1}{t^{k-1}}\cdot\cdot+A_{-1}\frac{1}{t}+A_0+\&c.,$$
and thence
$$\chi\varrho=\varrho A_{-s}.$$
The actual development, when k is small (for instance $k=1$ or $k=2$), is most readily obtained by developing each factor separately and taking the product. To do this we have
$$\frac{1}{1-ce^{-t}}=\frac{1}{1-c}-\frac{c}{(1-c)^2}t+\frac{c+c^2}{(1-c)^3}\tfrac{1}{2}t^2-\frac{c+4c^2+c^3}{(1-c)^4}\tfrac{1}{6}t^3+\&c.,$$
where by a general theorem for the expansion of any function of e^t, the coefficient of t^f is
$$=\frac{(-)^f}{\Pi f}\ \frac{1}{1-c(1+\Delta)}0^f$$
$$=\frac{(-)^f}{\Pi f}\left(\frac{1}{1-c}+\frac{c}{(1-c)^2}\Delta\ldots+\frac{c^f}{(1-c)^{f+1}}\right)0^f$$
(where as usual $\Delta 0^f=1^f-0^f$, $\Delta^2 0^f=2^f-2.1^f+0^f$, &c.) and
$$\frac{1}{1-e^{-t}}=\frac{1}{t}+\frac{1}{2}+\frac{1}{12}t-\frac{1}{720}t^3+\frac{1}{30240}t^5-\&c.,$$
where, except the constant term, the series contains odd powers only and the coefficient of t^{2f-1} is $\frac{(-)^{f+1}B_f}{\Pi 2f}$; B_1, B_2, B_3... denoting the series $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$... of BERNOULLI's numbers.

But when k is larger, it is convenient to obtain the development of the fraction from that of the logarithm, the logarithm of the fraction being equal to the sum of the logarithms of the simple factors, and these being found by means of the formulæ

$$\log\frac{1}{1-ce^{-t}}=\log\frac{1}{1-c}-\frac{c}{1-c}t+\frac{c}{(1-c)^2}\frac{t^2}{2}-\frac{c+c^2}{(1-c)^3}\frac{t^3}{6}+\frac{c+4c^2+c^3}{(1-c)^4}\frac{t^4}{24}+\&c.$$

$$\log\frac{1}{1-e^{-t}}=-\log t+\tfrac{1}{2}t-\frac{1}{24}t^2+\frac{1}{2880}t^4-\frac{1}{181440}t^6+\&c.$$

The fraction is thus expressed in the form

$$\frac{1}{\Pi(1-\varrho^n)\Pi(ap)}\frac{1}{t^k}e^{k_1t+k_2t^2+\dots};$$

and by developing the exponential we obtain, as before, the series commencing with $A_{-k}\frac{1}{t^k}$.

Resuming now the formula

$$\chi\varrho=\varrho A_{-s},$$

which gives $\chi\varrho$ as a function of ϱ, we have

$$\frac{\theta x}{[1-x^a]}=S\frac{\chi\varrho}{\varrho-x};$$

but this equation gives

$$\chi\varrho=\theta\varrho\left(\frac{\varrho-x}{[1-x^a]}\right)_{x=\rho},$$

and we have

$$[1-x^a]=(x-\varrho)(x-\varrho^{a_2})\dots(x-\varrho^{a_\alpha})$$

if 1, a_2, ... a_α are the integers less than a and prime to it (α is of course the degree of $[1-x^a]$). Hence

$$\chi\varrho=\theta\varrho\frac{-1}{\varrho^{\alpha-1}\Pi(1-\varrho^{a_i-1})},$$

and therefore

$$\theta\varrho=-\varrho^{\alpha-1}\Pi(1-\varrho^{a_i-1})\chi\varrho;$$

or putting for $\chi\varrho$ its value

$$\theta\varrho=-\varrho^{\alpha}\Pi(1-\varrho^{a_i-1})A_{-s},$$

where α is the degree of $[1-x^a]$ and a_i denotes in succession the integers (exclusive of unity) less than a and prime to it. The function on the right-hand, by means of the equation $[1-\varrho^a]=0$, may be reduced to an integral function of ϱ of the degree $\alpha-1$, and then by simply changing ϱ into x we have the required function θx. The fraction $\frac{\theta x}{[1-x^a]}$ can then by multiplication of the terms by the proper factor be reduced to a fraction with the denominator $1-x^a$, and the coefficients of the numerator of this fraction are the coefficients of the corresponding prime circulator ( )pcr a_q.

Thus, let it be required to find the terms depending on the denominator $[1-x^3]$ in

$$\frac{1}{(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)};$$

these are

$$\mathrm{S}\frac{\chi_1\rho}{\rho-x},\quad x\partial_x\,\mathrm{S}\frac{\chi_2\rho}{\rho-x},$$

where

$$\chi_1\rho=\text{coeff.}\,\frac{1}{t}\text{ in }\frac{\rho}{f(\rho e^{-t})}$$

$$\chi_2\rho=\text{coeff.}\,\frac{1}{t}\text{ in }t\frac{\rho}{f(\rho e^{-t})}$$

and

$$\frac{1}{f(\rho e^{-t})}=\frac{1}{(1-\rho e^{-t})(1-\rho^2e^{-2t})(1-\rho^4e^{-4t})(1-\rho^5e^{-5t})(1-e^{-3t})(1-e^{-6t})}$$

$$=\mathrm{A}_{-2}\frac{1}{t^2}+\mathrm{A}_{-1}\frac{1}{t}+\&\text{c.},$$

where it is easy to see that

$$\mathrm{A}_{-2}=\frac{1}{18}\,\frac{1}{(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)}$$

$$\mathrm{A}_{-1}=\frac{1}{(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)}\left\{\frac{1}{4}-\frac{1}{18}\left(\frac{\rho}{1-\rho}+\frac{2\rho^2}{1-\rho^2}+\frac{4\rho^4}{1-\rho^4}+\frac{4\rho^5}{1-\rho^5}\right)\right\},$$

and we have

$$\theta_2\rho=-\rho^2(1-\rho)\mathrm{A}_{-2}$$

$$\theta_1\rho=-\rho^2(1-\rho)\mathrm{A}_{-1}.$$

But $[1-\rho^3]=1+\rho+\rho^2=0$. Hence $\rho^3=1$, and therefore

$$(1-\rho)(1-\rho^2)(1-\rho^4)(1-\rho^5)=(1-\rho)^2(1-\rho^2)^2=9.$$

Hence

$$\theta_2\rho=-\frac{1}{162}\rho^2(1-\rho)=\frac{1}{162}(1-\rho^2)=\frac{1}{162}(2+\rho),$$

whence

$$\theta_2x=\frac{1}{162}(2+x),$$

and the partial fraction is

$$\frac{1}{162}\,\frac{2+x}{1+x+x^2},$$

which is

$$=\frac{1}{162}\,\frac{2-x-x^2}{1-x^3},$$

and gives rise to the prime circulator $\frac{1}{162}(2,\,-1,\,-1)$ pcr 3_q.

The reduction of $\theta_1\rho$ is somewhat less simple; we have

$$\theta_1\rho=-\frac{1}{9}\,\rho^2(1-\rho)\left\{\frac{1}{4}-\frac{1}{18}\left(\frac{\rho}{1-\rho}+\frac{2\rho^2}{1-\rho^2}+\frac{4\rho}{1-\rho}+\frac{5\rho^2}{1-\rho^2}\right)\right\}$$

$$=-\frac{1}{9}\,\rho^2(1-\rho)\left\{\frac{1}{4}-\frac{5}{18}\,\frac{\rho}{1-\rho}-\frac{7}{18}\,\frac{\rho^2}{1-\rho^2}\right\}$$

$$=\ \frac{1}{9}(1-\rho^2)\left\{\frac{1}{4}-\frac{5}{54}\rho(1-\rho^2)-\frac{7}{54}\rho^2(1-\rho)\right\}$$

$$= \frac{1}{972}(1-\varrho^2)(51-10\varrho-14\varrho^2)$$

$$= \frac{1}{972}(61+4\varrho-65\varrho^2);$$

whence, finally,

$$\theta_1\varrho=\frac{1}{324}(42+23\varrho),\quad \theta_1x=\frac{1}{324}(42+23x);$$

and the partial fraction is

$$\frac{1}{324}x\partial_x\frac{42+23x}{1+x+x^2},$$

which is

$$=\frac{1}{324}x\partial_x\frac{42-19x-23x^2}{1-x^3},$$

and gives rise to the prime circulator $\frac{1}{324}q(42,\ -19,\ -23)$ pcr 3_q.

The part depending on the denominator $1-x$ is

$$\frac{A_{-1}}{1-x}+\frac{1}{1}x\partial_x\frac{A_{-2}}{1-x}+\frac{1}{1.2}(x\partial_x)^2\frac{A_{-3}}{1-x}\cdots+\frac{1}{1.2.3.4.5}(x\partial_x)^5\frac{A_{-6}}{1-x},$$

where

$$\frac{1}{(1-e^{-t})(1-e^{-2t})(1-e^{-3t})(1-e^{-4t})(1-e^{-5t})(1-e^{-6t})}$$

$$=A_{-6}\frac{1}{t^6}+A_{-5}\frac{1}{t^5}\cdots+A_{-1}\frac{1}{t}+\text{ \&c.}$$

We have here

$$\log\frac{1}{1-e^{-t}}=-\log t+\frac{1}{2}t-\frac{1}{24}t^2+\frac{1}{2880}t^4-\text{ \&c.},$$

and thence the fraction is

$$\frac{1}{720\,t^6}\,e^{\frac{21}{2}t-\frac{91}{24}t^2+\frac{455}{576}t^4-\text{ \&c.}}$$

which is equal to

$$\frac{1}{720\,t^6}\left(1+\frac{21}{2}t+\frac{441}{8}t^2+\frac{3087}{16}t^3+\frac{64827}{128}t^4+\frac{1361367}{1280}t^5+\cdots\right.$$

$$\times\left(1-\frac{91}{24}t^2+\frac{8281}{1152}t^4+\cdots\right)$$

$$\times\left(1+\frac{455}{576}t^4+\cdots\right)$$

$$=\frac{1}{720}\frac{1}{t^6}+\frac{7}{480}\frac{1}{t^5}+\frac{77}{1080}\frac{1}{t^4}+\frac{245}{1152}\frac{1}{t^3}+\frac{43981}{103680}\frac{1}{t^2}+\frac{199577}{345600}\frac{1}{t}+\cdots$$

and consequently the partial fractions are

$$\frac{1}{86400}(x\partial_x)^5\frac{1}{1-x}+\frac{7}{11520}(x\partial_x)^4\frac{1}{1-x}+\frac{77}{6480}(x\partial_x)^3\frac{1}{1-x}+\frac{245}{2304}(x\partial_x)^2\frac{1}{1-x}$$

$$+\frac{43981}{103680}(x\partial_x)\frac{1}{1-x}+\frac{199577}{345600}\frac{1}{1-x},$$

from which the non-circulating part is at once obtained.

The complete expression for the number of partitions is $P(1, 2, 3, 4, 5, 6)q=$

$$\frac{1}{1036800}(12q^5+630q^4+1230q^3+110250q^2+439810q+598731)$$

$$+\frac{1}{4608}(6q^2+126q+581)(1, -1)\operatorname{pcr} 2_q$$

$$+\frac{1}{162}q \ldots\ldots\ldots (2, -1, -1)\operatorname{pcr} 3_q$$

$$+\frac{1}{324} \ldots\ldots (42, -19, -23)\operatorname{pcr} 3_q$$

$$+\frac{1}{32} \ldots\ldots (1, 1, -1, -1)\operatorname{pcr} 4_q$$

$$+\frac{1}{25} \ldots\ldots (2, 1, 0, -1, -2)\operatorname{pcr} 5_q$$

$$+\frac{1}{36} \ldots (2, 1, -1, -2, -1, 1)\operatorname{pcr} 6_q.$$